International Conference on Computational & Experimental Engineering and Sciences. Corfu, Greece, July 2003.

Dimensional Analysis of Yielding Structures

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Summary

This paper summarizes recent work by the authors that revisits the way of presenting information on the dynamic response of inelastic structures. The nonlinear response of rigid-plastic and elastic-plastic systems is presented in terms of the dimensionless Π-products that result from rigorous dimensional analysis. The main advantage of the analysis presented in this study is that it brings forward the concept of self-similarity—an invariance with respect to changes in scale or size—which is a decisive symmetry that shapes nonlinear behavior.

Introduction

Within the context of earthquake engineering the first systematic work on the response of an elastic-plastic single-degree-of-freedom system subjected to earthquake and pulse-type ground shaking was presented in the seminal papers by Veletsos and Newmark[1] and Veletsos et al.[2]. In these pioneering studies the response of the elastic-plastic system was normalized to the response of an elastic system having the same stiffness as the initial stiffness of the inelastic system. This approach was mainly motivated from: (a) the need to explain why the forces that develop in yielding structures are considerably smaller than the forces computed from elastic analysis; and (b) an idea that the energy input in the two systems should be comparable.

In this paper we first present an alternative way of presenting the nonlinear response of an elastic-plastic system which is derived from formal dimensional analysis (Langhaar[3], Housner and Hudson[4], Barenblatt[5]). The proposed dimensionless variables are liberated from the associated elastic system response and reveal remarkable order in the normalized response. It is most interesting, that the fundamental concepts upon which the proposed dimensional analysis builds have been put forward in the 1965 Newmark's Rankine lecture[6] and in the paper by Veletsos et al.[2].

Our interest in this study focusses on the response of yielding structures under strong earthquake shaking which is the strongest nearby the causative faults where in most occasions the kinematic characteristics of the ground exhibit distinguishable pulses. Accordingly, our investigation focusses on the response analysis of elastoplastic and bilinear single-degree-of-freedom oscillators subjected to pulse-type excitations. The ability of distinct pulses to generate structural response that resembles the earthquake induced response has been examined in past studies (Veletsos et al.[2], Yim et al. [7], Hall et al.[8], Makris and Chang[9], Makris and Roussos[10], Chang et al.[11] among others). A recent study on the nonlinear response of frame structures subjected to near-source ground motions has been conducted by Alavi and Krawinkler[12]. Pulse-type excitations are of key importance in this study because they allow the introduction of dimensionless parameters which uncover the underlying physics of the response.

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Dimensional Analysis of Rigid Plastic and Elastoplastic Systems

Within the context of earthquake engineering an early solution to the response of a rigid-plastic system (rigid mass sliding on a moving base—see Figure 1) subjected to a rectangular acceleration pulse has been presented by Newmark[6]. In this case, the strength of a rigid-plastic system is $Q = \mu m g$. Under a rectangular acceleration pulse with amplitude $a_p > \mu g$ and duration, T_p,

$$\ddot{u}_g(t) = a_p, \qquad 0 \le t \le T_p \tag{1}$$

the entire relative displacement of the mass on the moving surface is (Newmark [6])

$$u_{max} = \frac{a_p T_p^2}{2}\left(\frac{a_p}{\mu g} - 1\right) = \frac{a_p T_p^2}{2}\left(\frac{m a_p}{Q} - 1\right). \tag{2}$$

Equation (2) indicates that the plastic displacement is proportional to the intensity of the acceleration pulse, a_p, and the square of its duration, T_p^2. The product, $a_p T_p^2 / 2 \approx L_e$ is a characteristic length scale (in this case, the displacement of the base when the acceleration pulse expires) of the ground excitation and is a measure of the intensity of the excitation pulse. The rectangular acceleration pulse used by Newmark[1] which leads to an infinite base displacement is probably the most well-suited example to introduce the finite length scale, $L_e = a_p T_p^2$, of the energetic pulse of the motion, that eventually leads to an infinite displacement. Upon the expiration of the pulse, the base moves with a constant velocity and the inertia demand on the structure is zero. This situation is reminiscent of the minor seismic demands on structures subjected to selected near-source ground motions which upon the expiration of the main pulse, the earthquake induces incrementally very large ground displacements with feeble inertia effects.

Following this discussion it is natural to normalize the relative structural response, u_{max}, to the length scale of the energetic excitation, L_e, and Equation (2), is re-written as

$$\frac{u_{max}}{a_p T_p^2} = \frac{1}{2}\left(\frac{m a_p}{Q} - 1\right). \tag{3}$$

Equation (3), which was obtained by solving the differential equation that governs the sliding response, relates the dimensionless displacement $\Pi_1 = u_{max} / a_p T_p^2$ to the dimensionless strength $\Pi_2 = Q / m a_p$. This relation,

$$\Pi_1 = \frac{1}{2}\left(\frac{1}{\Pi_2} - 1\right), \tag{4}$$

is plotted with a heavy line in Figure 2 (left) in a logarithmic scale.

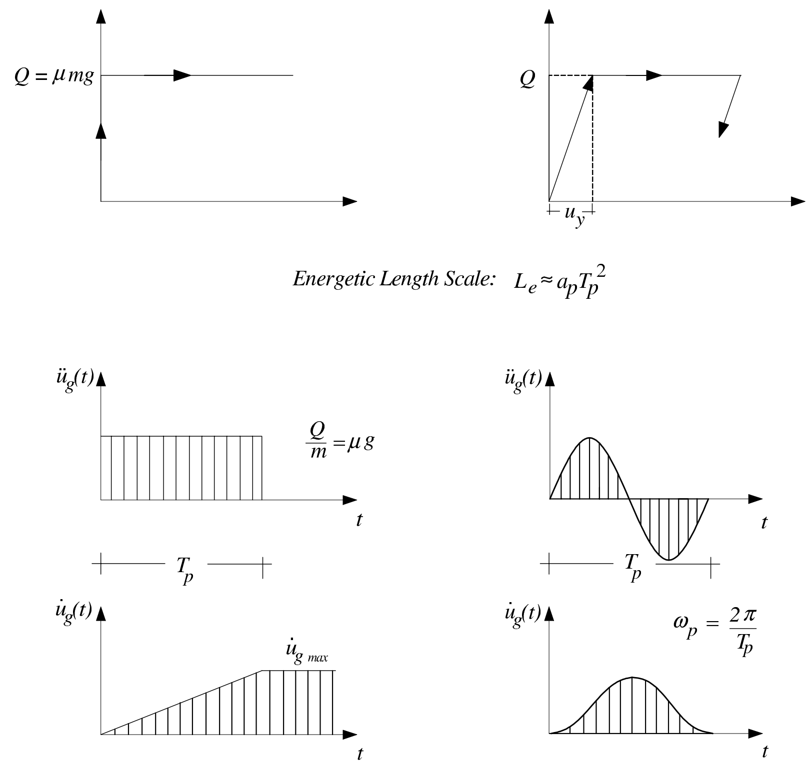

Figure 1. Rigid-plastic and elastic-plastic behavior (top); and the acceleration and velocity time histories of a rectangular and a one-sine acceleration pulse (bottom).

Dimensional analysis is a mathematical tool that shapes the general form of relations that describe natural phenomena. The application of dimensional analysis to any particular physical phenomenon is based on the premise that the phenomenon can be described by a dimensionally homogeneous equation that relates the dependent variables, u, and the independent variables, $u_2, \ldots, u_k$:

$$u_1 = f(u_2, u_3, \ldots, u_k) . \tag{5}$$

For instance, in the case of a rigid-plastic system subjected to an acceleration pulse with amplitude, a_p, and duration, T_p, it is expected that the maximum relative displacement, u_{max}, is a function of the specific strength of the system, $Q/m = \mu g$, and the characteristics of the pulse, a_p and T_p giving

$$u_{max} = f\left(\frac{Q}{m}, a_p, T_p\right) . \tag{6}$$

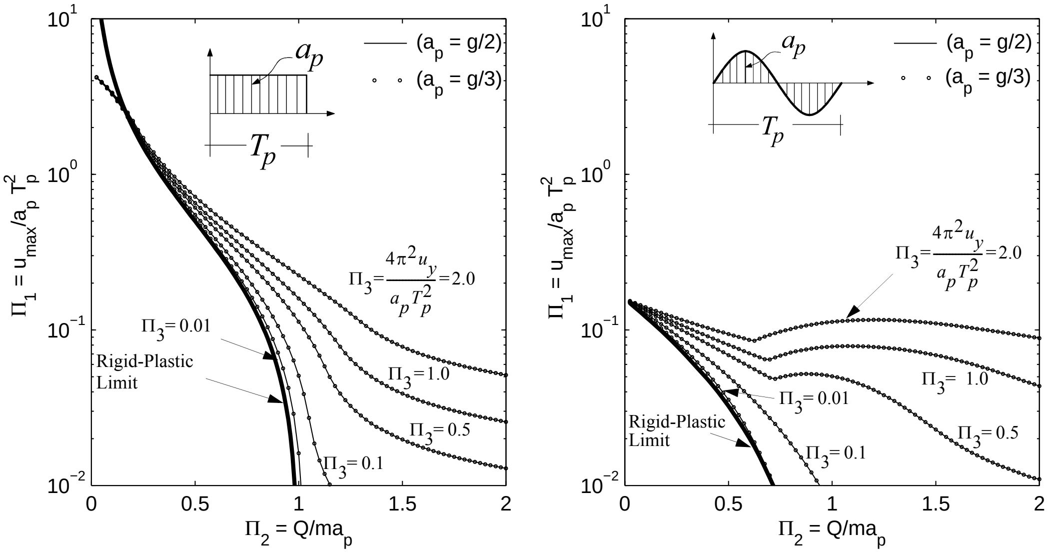

Figure 2. Normalized relative displacement curves of an elastic-plastic structure subjected to a rectangular and a one-sine acceleration pulses. The self-similar solutions approach the rigid-plastic limit as the normalized yield displacement tends to zero.

In the case of Equation (6), the four variables $u_{max} \doteq [L]$, $Q/m \doteq [L][T]^{-2}$, $a_p \doteq [L][T]^{-2}$ and $T_p \doteq [T]$ involve only two reference dimensions ($r = 2$), that of length $[L]$ and time $[T]$. According to Buckingham's Π-theorem the number of independent dimensionless Π-products is equal to the number of physical variables appearing in Equation (6) (4 variables) minus the number of reference dimensions (two). Therefore, for a rigid-plastic system subjected to an acceleration pulse we have 4-2=2 Π-terms. Since the repeating variables need to have independent dimensions, the obvious choice for the repeating variables is the acceleration amplitude of the pulse, a_p, and its duration, T_p, which gives $\Pi_1 = u_{max}/a_p T_p^2$ and $\Pi_2 = Q/ma_p$. With the two Π-terms established, Equation (6) reduces to

$$\frac{u_{max}}{a_p T_p^2} = \phi\left(\frac{Q}{ma_p}\right). \tag{7}$$

In the elementary case of a rectangular acceleration pulse the form of the function ϕ was obtained analytically by solving the differential equations and is given by Equation (3). For trigonometric pulses such as a one sine pulse or a one cosine pulse introduced earlier, the response of the rigid-plastic system is also described by Equation (7) and the form of the function ϕ is obtained numerically. Figure 2 (right) plots with a heavy line the response of the rigid-plastic system when subjected to a one-sine acceleration (Type-A) pulse. The response is plotted on a logarithmic scale next to the response from a rectangular pulse in order to illustrate the relative strength of a rectangular acceleration pulse and a forward displacement (one-sine acceleration pulse).

The idealized rigid-plastic system analyzed in the preceding section exhibits zero yield displacement (infinite preyielding stiffness), and therefore infinite ductility. We now consider the response of an inelastic system that exhibits a finite yield displacement before sliding. With reference to Figure 1 (right) the response of an elastic-plastic system subjected to some acceleration pulse of amplitude a_p and duration T_p, should be a function of the specific strength, Q/m, the yield displacement, u_y, and the characteristics of the pulse, a_p and $\omega_p = 2\pi/T_p$. Accordingly,

$$u_{max} = f\left(\frac{Q}{m}, u_y, a_p, \omega_p\right) . \tag{8}$$

The five variables appearing in (8) involve only two reference dimensions that of length $[L]$ and time $[T]$. According to Buckingham's Π-theorem the number of independent dimensionless Π-products is now: (5 variables) - (2 reference dimensions) = 3 Π-terms.

$$\frac{u_{max}\omega_p^2}{a_p} = \phi\left(\frac{Q}{ma_p}, \frac{u_y\omega_p^2}{a_p}\right) , \tag{9}$$

Figure 2 illustrates how the response of the elastic-plastic system amplifies as the normalized yield displacement increases. The most notable observation is that the normalized response is invariant with respect to the level of the acceleration amplitude, a_p—the solid and dotted lines have been computed for different levels of acceleration, a_p, yet the normalized response is identical. This scale invariance between the size of the maximum relative displacement, the size of the yield displacement, and the intensity of the acceleration pulse is known as self-similarity (Langhaar[3], Barenblatt[5]) which is a special type of symmetry that has unique importance in understanding and ordering nonlinear response. Another interesting observation is that for values of normalized strength below one ($\Pi_2 = Q/ma_p \le 1$), the rectangular pulse induces much larger displacements than the one-sine acceleration pulse; whereas, as the value of the normalized strength, Q/ma_p increases the situation reverses.

Conclusion

When the response of inelastic structures is presented in terms of the dimensionless Π-terms the response curves are self-similar and follow a single master curve. This remarkable order in the response is invariant with respect to changes in scale or size. The dimensional analysis presented in this study shows that what really matters when ordering inelastic response, is not the yield displacement, u_y, alone but its normalized value to the energetic length scale of the excitation, $L_e \approx a_p/\omega_p^2$. The self-similar solutions derived for trigonometric pulses considered herein show that:

- For small values of the normalized strength, $\Pi_2 = Q/ma_p$, the normalized displacement, $\Pi_1 = u_{max}\omega_p^2/a_p$, is nearly independent of the normalized yield displacement, $\Pi_3 = u_y\omega_p^2/a_p$
- For larger values of Π_2 the response depends strongly on Π_3. Most interestingly, there is a strength range where an increase in strength results in an increase in displacements—a counter intuitive situation.

Additional results are available in Makris and Black[13].

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